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COMPUTATIONAL ASPECTS OF CLOSED QUEUEING NETWORKS WITH DIFFERENT CUSTOMER CLASSES[1]

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Abstract: Networks of queues are important models of multiprogrammed and time-shared computer systems. We consider the computational aspects of a class of networks studied by Baskett, Chandy, Muntz, and Palacios [BCMP75]. We find that the computational algorithms employed by Chandy, Herzog, and Woo [CHW75] for closed queueing networks with different classes of customers without class switching can be easily extended to networks which do allow customers to switch class. Expressions derived for the throughput of a queue are exceedingly simple.

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Introduction

We will consider a class of closed queueing networks studied by [BCMP75]. Our objective is to present clear, concise, and computationally efficient algorithms for the calculation of:

(1) the "normalization constant",

(2) the marginal queue length distribution,

(3) server utilizations,

(4) the throughput and mean response (or waiting) time of each service center, and

(5) the queue length distribution at arrival.

We will present these algorithms for closed networks with only one class of customers first and then show how they can be extended to multiple job class networks. This paper draws heavily on the ideas and results of [CHW75] and may in fact be viewed as an attempt to clarify and extend many of these ideas.

Network Description (Single Customer Class)

This section presents notational conventions employed in specifying the closed queueing network model. The general model consists of a finite number, M, of service centers indexed 1,2,...,M and a finite number, N, of customers. (We will use the words service center, station, and queue interchangeably). The movement of customers in the network is governed by an M x M transition matrix $P = [p_{ij}]$ where p_{ij} is the probability of a customer, after completing service at service center i, will next move to service center j. The states of this network are M-tuples $(n_1, n_2, \ldots, n_i, \ldots, n_M)$ where n_i is the number of customers at service center i, $0 \le n_i \le N$, $1 \le i \le M$. The state is termed feasible if in addition

$$\sum_{i=1}^{M} n_i = N$$

We will denote a feasible state by $\underline{n} = (n_1, n_2, \ldots, n_i, \ldots, n_M)$ and the set of all feasible states by

$$S(N,M) = \{ (n_1, n_2, \ldots, n_i, \ldots, n_M) \mid 0 \le n_i \le N \text{ for } 1 \le i \le M \text{ and } \sum_{i=1}^{M} n_i = N \}$$

Note that the transition probabilities are independent of the state of the network. Let $\mu_i(k)$ denote the mean service rate of service center i when there are k (=1,2,...,N) customers at that service center. Each service center is further characterized by its service discipline. The following service center types are permitted:

(1) Single server, First-Come-First-Serverd (FCFS) queueing discipline, and exponential service time distribution.

(2) Single server, Processor Sharing (PS) queueing discipline (i.e. when there are n customers in the service center, each customer is receiving service at a rate of 1/n sec. per sec.). General service time distributions are permitted under the restriction that they have rational Laplace transforms.

(3) Infinite servers (IS), so that no customer waits for service. General service time distributions are permitted under the restriction that they have rational Laplace transforms.

(4) Single server, Last-Come-First-Served Preemptive Resume (LCFS) queueing discipline. General service time distributions are permitted under the restriction that they have rational Laplace transforms.

Note: multiple server stations can be modelled using load dependent service rates. For example, a station with n servers can be modelled as a service center with a load dependent service rate given by

$$\mu_i(k) = \begin{cases} k\mu_i(1) \text{ if } k \le n \\ n\mu_i(1) \text{ if } k > n \end{cases}$$

Joint Probability Distribution (Single Customer Class)

Let $p(n_1,n_2,\ldots,n_M)$ be the probability that the system is in state $(n_1,n_2,\ldots,n_M)$ which is assumed to be a feasible state. In [BCMP75] it is shown that for closed queueing networks with service centers of types 1, 2, 3, or 4, the equilibrium state probabilities have the product form:

$$p(n_1,n_2,\ldots,n_M) = \frac{1}{G(N)} \prod_{i=1}^{M} f_i(n_i) \qquad (1)$$

where

$$f_i(n_i) = \begin{cases} \dfrac{e_i^{n_i}}{\prod_{j=1}^{n_i} \mu_i(j)} & \text{if service center i is of type 1, 2, or 4} \\[2ex] \dfrac{1}{n_i!}\left(\dfrac{e_i}{\mu_i}\right)^{n_i} & \text{if service center i is of type 3} \end{cases} \qquad (2)$$

G(N) is the normalizing constant defined so that all the equilibrium state probabilities sum to 1,

and

e_i , $1 \le i \le M$ are solutions to the following set of M linear equations:

$$e_j = \sum_{i=1}^{M} e_i p_{ij} \quad , \qquad j=1,2,\ldots,M$$

(Note that the set of numbers e_i are unique up to a multiplicative constant). The normalization constant G(N) must be evaluated in order to obtain numerical values of the state probabilities. Unfortunately, if G(N) is computed by application of its definition, the computation will grow combinatorially with the number of states. For this reason alternate methods have been devised to evaluate the normalization constant G(N).

Computation of G(N) (Single Customer Class)

We will now develop an efficient iterative algorithm for the computation of the normalization constant G(N) introduced by [BUZ71, BUZ73]. The algorithm actually computes the entire set of values G(1), G(2),...,G(N).

The first step is to note as above that the equilibrium state probabilities must sum to 1. Hence,

$$1 = \sum_{\underline{n} \in S(N,M)} p(n_1,n_2,\ldots,n_M)$$

$$= \frac{1}{G(N)} \sum_{\underline{n} \in S(N,M)} \prod_{i=1}^{M} f_i(n_i)$$

and
$$G(N) = \sum_{\underline{n} \in S(N,M)} \prod_{i=1}^{M} f_i(n_i) \tag{3}$$

In order to derive the algorithm, we define an auxiliary function identical to [BUZ73]. Let

$$G_m(n) = \sum_{\underline{n} \in S(n,m)} \prod_{i=1}^{m} f_i(n_i) \tag{4}$$

Note that G(N) as defined by (3) is equal to $G_M(N)$ and, in fact, $G_M(n) = G(n)$ for n=0,1,...,N. Hence, observe that $G_M(n)$ is the normalization constant for a closed network with n customers. Now for m > 1,

$$G_m(n) = \sum_{n \in S(N,M)} \prod_{i=1}^{m} f_i(n_i)$$

$$= \sum_{j=0}^{n} \Big(\sum_{\substack{\underline{n} \in S(n,m) \\ n_m = j}} \prod_{i=1}^{m} f_i(n_i) \Big)$$

$$= \sum_{j=0}^{n} f_m(j) \cdot \sum_{\underline{n} \in S(n-j,m-1)} \prod_{i=1}^{m-1} f_i(n_i)$$

$$= \sum_{j=0}^{n} f_m(j)\; G_{m-1}(n-j) \tag{5}$$

We then see that to compute $G_m(n)$, n=0,1,...,N we must first compute $G_{m-1}(j)$, j=0,1,...,n. The iterative computation (5) must be initialized by computing $G_1(n)$, n=0,1,...,N. But this follows easily from the defining relation (4), i.e.,

$$G_1(n) = \sum_{\underline{n}\in S(n,1)} \prod_{i=1}^{1} f_i(n_i)$$

$$= \sum_{\underline{n}\in\{(n)\}} f_i(n)$$

$$= f_1(n) \qquad \text{for } n=0,1,\ldots,N \tag{6}$$

The iterative relationship defined by (5), together with the initial conditions specified in (6), completely defines the algorithm to compute $G_m(n)$ and hence $G_M(N)$.

Notice that the sum in eq. (5) is similar to a discrete case convolution. We will exploit this idea in order to obtain a compact notational device for the algorithms yet to be presented. We define a "convolution" between two L+1 dimensional vectors

$$\underline{A} = (A(0),A(1),\ldots,A(L)) \quad \text{and} \quad \underline{B} = (B(0),B(1),\ldots,B(L))$$

as an L+1 dimensional vector

$$\underline{C} = (C(0),C(1),\ldots,C(L))$$

where

$$C(n) = \sum_{j=0}^{n} A(j)\,B(n-j) \qquad \text{for } n=0,1,\ldots,L$$

and denote the vector operation as *. Thus,

$$\underline{C} = \underline{A} * \underline{B}.$$

Next, we will cast equations (5) and (6) into our new notation. We define an N+1 dimensional vector

$$\underline{f_i} = (f_i(0), f_i(1), \ldots, f_i(N))$$

where $f_i(0)$ is given by (2). We define M vectors

$$\underline{G_1}\ ,\ \underline{G_2}\ ,\ \ldots\ ,\ \underline{G_M}$$

each of dimension N+1 where

$$\underline{G_1} = (G_1(0), G_1(1), \ldots, G_1(N))$$

$$= (f_1(0), f_2(0), \ldots, f_1(N)) \quad \text{(cf. eq. (6))}$$

$$= \underline{f_1}$$

and $\underline{G_m} = \underline{f_m} * \underline{G_{m-1}}$ for $m=2,3,\ldots,M$ (cf. eq. (5)).

Marginal Queue Length Distribution (Single Customer Class)

We will now cover the computation of the marginal queue length distribution. Let $p_i(n)$ be the marginal probability that there are n customers at service center i. Thus, for $n=0,1,\ldots,N$,

$$p_i(n) = \sum_{\substack{\underline{n} \in S(N,M) \\ n_i = n}} p(\underline{n})$$

$$= \sum_{\substack{n_1+\cdots+n_{i-1}+n+n_{i+1}+\cdots+n_M=N \\ n_j \geq 0,\ j=1,2,\ldots,M}} p(n_1,\ldots,n_{i-1},n,n_{i+1},\ldots,n_M)$$

$$= \frac{f_i(n)}{G(N)} \sum_{\substack{\sum_{j=1, j\neq i}^{M} n_j = N-n \\ n_j \geq 0,\ j=1,2,\ldots,M}} \prod_{\substack{k=1 \\ k\neq i}}^{M} f_k(n_k) \tag{7}$$

Before proceeding we redefine our auxiliary function in order to delete a specific service center from attention, i.e. let

$$G_m^i(n) = \sum_{\substack{\sum_{j=1, j\neq i}^{m} n_j = n \\ n_j \geq 0,\ j=1,2,\ldots,m}} \prod_{\substack{k=1 \\ k\neq i}}^{m} f_k(n_k) \tag{8}$$

Note carefully that this and the previous definition of the auxiliary function (4) are related by

$$G_{M-1}(n) = G_{M-1}^{M}(n) = G_M^{M}(n) \qquad \text{for } n=0,1,\ldots,N$$

Now substituting our new auxiliary function definition (8) into (7) we arrive at

$$p_i(n) = \frac{f_i(n)}{G(N)}\, G_M^i(N-n) \qquad \text{for } n=0,1,\ldots,N \tag{9}$$

We can readily see that in order to compute the marginal probability distribution we need a procedure to evaluate the auxiliary function (8).

Fortunately, a simple recurrence relation for computing (8) exists. The derivation of this formula uses the fact that the marginal probabilities must sum to 1. Assume that the closed queueing network has a fixed number of customers, K, K=0,1,...,N. Then

$$1 = \sum_{n=0}^{K} p_i(n) \qquad \text{for any } i,\ 1 \le i \le M$$

$$= \sum_{n=0}^{K} \frac{f_i(n)}{G(K)} G_M^i(K-n)$$

and

$$G(K) = \sum_{n=0}^{K} f_i(n)\, G_M^i(K-n) \tag{10}$$

$$= f_i(0)\, G_M^i(K) + \sum_{n=1}^{K} f_i(n)\, G_M^i(K-n) \qquad k=1,2,\ldots,N$$

Noting from eq. (2) that $f_i(0) = 1$, the desired recurrence relation becomes

$$G_M^i(K) = G(K) - \sum_{n=1}^{K} f_i(n)\, G_M^i(K-n) \tag{11}$$

Notice that (11) is a recurrence relation in K. The values of

$$G_M^i(K)$$

are computed recursively starting with K=1 and the initial condition derived from eqns. (3) and (10):

$$1 = G(0) = f_i(0)\, G_M^i(0) = G_M^i(0)$$

Equation (11) then defines the components of the vector

$$G_M^i$$

which are required in eq. (9).

Service Center Utilization (Single Customer Class)

Let U_i be the fraction of time service center i is busy. This is easily computed once we realize that the utilization of a service center is just the probability that it is active, i.e. that there are one or more customers at the service center. Hence,

$$U_i = \sum_{n=1}^{N} p_i(n)$$

$$= 1 - p_i(0)$$

$$= 1 - \frac{f_i(0)}{G(N)} G_M^i(N)$$

$$= 1 - \frac{G_M^i(N)}{G(N)}$$

Throughput and Mean Waiting Time (Single Customer Class)

Let $TPUT_i$ be the throughput of service center i, i.e. the rate at which customers get serviced and leave the service center. We have two cases depending upon the service center type:

For type 1,2, or 4:

$$TPUT_i = \sum_{n=1}^{N} p_i(n)\mu_i(n)$$

$$= \sum_{n=1}^{N} \frac{f_i(n)}{G(N)} G_M^i(N-n)\ \mu_i(n)$$

$$= \sum_{n=1}^{N} \frac{e_i}{\mu_i(n)} \frac{f_i(n-1)}{G(N)} G_M^i(N-n)\ \mu_i(n) \qquad \text{(cf. eq. 2)}$$

$$= \frac{e_i}{G(N)} \sum_{n=1}^{N} f_i(n-1)\ G_M^i(N-n)$$

$$= \frac{e_i}{G(N)} \sum_{j=0}^{N-1} f_i(j)\ G_M^i(N-1-j)$$

$$= \frac{e_i}{G(N)} G(N-1) \qquad \text{(cf. eq. 10)}$$

For a type 3 service center:

$$TPUT_i = \sum_{n=1}^{N} p_i(n)\ n\ \mu_i$$

$$= \sum_{n=1}^{N} \frac{f_i(n)}{G(N)} G_M^i(N-n)\ n\ \mu_i$$

$$= \sum_{n=1}^{N} \frac{e_i}{n\mu_i} \frac{f_i(n-1)}{G(N)} G_M^i(N-n)\ n\ \mu_i$$

$$= \frac{e_i}{G(N)} \sum_{n=1}^{N} f_i(n-1)\ G_M^i(N-n)$$

$$= \frac{e_i}{G(N)} \sum_{j=0}^{N-1} f_i(j)\ G_M^i(N-1-j)$$

$$= \frac{e_i}{G(N)}\ G(N-1)$$

Hence, we have the same expression for throughput regardless of service center type.

Let W_i be the mean waiting time of service center i, i.e. the average time a customer spends at service center i (queueing time plus service time). This can be determined using Little's formula [LIT61],

$$W_i = \bar{n}_i \ / \ \lambda_i$$

where $\bar{n}_i$ is the average number of customers at service center i (i.e. the mean queue length) and λ_i is the average arrival rate at service center i, when the system is in equilibrium. But the throughput of a service center is equal to the arrival rate to that service center if the system is in equilibrium. Hence, $\lambda_i = TPUT_i$ for $i=1,2,\ldots,M$. Also, $\bar{n}_i$ may be computed as

$$\bar{n}_i = \sum_{n=1}^{N} n\ p_i(n)$$

Therefore, $T_i = \bar{n}_i / TPUT_i$.

Queue Length Distribution (Single Customer Class)

Let $Q_i(n)$ be the probability that n-1 customers are already queued (or in service) at station i on the arrival of the nth customer (n=1,2,...,N). In order to determine $Q_i(n)$ we first find the net rate at which customers enter the ith queue when there are n-1 customers already there. Let

$$\underline{n}_{ij} = (n_1, n_2, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_M)$$

and

$$\underline{n} = (n_1, n_2, \ldots, n_i, \ldots, n_j, \ldots, n_M)$$

be feasible states with $i \neq j$. The net rate at which the system transits from state $\underline{n}_{ij}$ to state $\underline{n}$ due to a customer arriving at station i after finishing service at station j is:

$$p(\underline{n}_{ij})\ \mu_j(n_j+1)\ p_{ij}$$

$$= \frac{1}{G(N)} f_1(n_1) \cdots f_i(n_i-1) \cdots f_j(n_j+1) \cdots f_M(n_M) \mu_j(n_j+1) p_{ij}$$

$$= \frac{1}{G(N)} f_1(n_1) \cdots f_i(n_i-1) \cdots \frac{e_j}{\mu_j(n_j+1)} f_j(n_j) \cdots f_M(n_M)\ \mu_j(n_j+1)\ p_{ij}$$

$$= \frac{1}{G(N)} f_1(n_1) \cdots f_i(n_i-1) \cdots f_j(n_j) \cdots f_M(n_M)\ e_j\ p_{ij}$$

Defining the filter function

$$\beta(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{otherwise} \end{cases}$$

the total net rate at which state $\underline{n}$ is entered due to an arrival at queue i becomes

$$r_i(\underline{n}) = \sum_{\substack{j=1 \\ j\neq i}}^{M} p(\underline{n}_{ij})\mu_j(n_j+1)p_{ji}\beta(n_i) + p(\underline{n})\mu_i(n_i)p_{ii}\beta(n_i)$$

$$= \sum_{\substack{j=1 \\ j\neq i}}^{M} \frac{1}{G(N)} f_1(n_1)\cdots f_i(n_i-1)\cdots f_j(n_j)\cdots f_M(n_M)e_j p_{ji}\beta(n_i)$$

$$+ \frac{1}{G(N)} f_1(n_1)\cdots\frac{e_i}{\mu_i(n_i)} f_i(n_i-1)\cdots f_j(n_j)\cdots f_M(n_M)\mu_i(n_i)p_{ii}\beta(n_i)$$

$$= \frac{\beta(n_i)}{G(N)} f_1(n_1)\cdots f_i(n_i-1)\cdots f_j(n_j)\cdots f_M(n_M) \sum_{j=1}^{M} e_j \ p_{ji}$$

$$= \frac{\beta(n_i)}{G(N)} f_1(n_1)\cdots f_i(n_i-1)\cdots f_j(n_j)\cdots f_M(n_M) \ e_i$$

We can now derive the conditional probability of a transition into state $\underline{n}$ by the addition of one customer in queue i. This is accomplished by computing the frequency of such transitions over a time interval of length T. Let Y_i be the number of transitions into state $\underline{n}$ by adding a customer in queue i during the interval T. Then by the definition of rate

$$Y_i = r_i(\underline{n}) \ T$$

Note that since the choice of the ith queue in the previous derivations is completely arbitrary, the total number of transitions into state $\underline{n}$ is simply obtained by summing over all possible transitions into state $\underline{n}$ from any queue. Thus the (relative) frequency with which such a transition occurs is

$$\psi_i(\underline{n}) = \frac{Y_i}{\sum_{j=1}^{M} Y_j}$$

Therefore, if we define $\varphi_i(n)$ to be the probability of entering state $\underline{n}$ adding one customer to queue i we get

$$\varphi_i(\underline{n}) = \lim_{T\to\infty} \psi_i(\underline{n}) = \lim_{T\to\infty} \frac{Y_j}{\sum_{j=1}^{M} Y_j}$$

$$= \lim_{T\to\infty} \frac{r_i(\underline{n})\ T}{\sum_{j=1}^{M} r_j(\underline{n})\ T} = \lim_{T\to\infty} \frac{r_i(\underline{n})}{\sum_{j=1}^{M} r_j(\underline{n})}$$

Let us define $\rho_i(n)$ as the net rate at which customers enter queue i when there are n-1 customers already present there, without imposing any restriction upon the population of the other queues (other than insuring having a feasible state).

The state $\underline{n}$ in the previous derivations is really one of many states satisfying the constraint that there are n customers at queue i. Therefore, we can repeat the computation of the rate $r_i(\underline{n})$ for any state $\underline{n}$ such that

$$\underline{n} \in \{ (n_1, n_2, \ldots, n_i, \ldots, n_M) \mid \sum_{\substack{j\neq i \\ j=1}}^{M} n_j = N-n \}$$

Evidently any one of these rates is a component of $\rho_i(n)$, so that we can write:

$$e_i(n) = \sum_{\substack{\underline{n} \in \{(n_1,\ldots,n_i,\ldots,n_M) \mid \\ \sum_{j=1, j\neq i}^{M} n_j = N-n\}}} r_i(\underline{n})$$

$$= \sum_{\substack{\sum_{j=1, j\neq i}^{M} n_j = N-n}} \frac{1}{G(N)} f_1(n_1) f_2(n_2) \cdots f_i(n-1) \cdots f_M(n_M) e_i$$

$$= \frac{e_i f_i(n-1)}{G(N)} \sum_{\substack{\sum_{j=1, j\neq i}^{M} n_j = N-n}} f_1(n_1) \cdots f_{i-1}(n_{i-1}) f_{i+1}(n_{i+1}) \cdots f_M(n_M)$$

$$= \frac{e_i f_i(n-1) G_M^i(N-n)}{G(N)} \qquad \text{(cf. eq. 8)}$$

$$= \frac{f_i(n) G_M^i(N-n)}{G(N)} \frac{f_i(n-1)}{f_i(n)} e_i$$

$$= p_i(n) \frac{f_i(n-1)}{f_i(n)} e_i$$

$$= \frac{p_i(n) \; f_i(n-1) \; e_i}{\dfrac{e_i \; f_i(n-1)}{\mu_i(n)}}$$

$$= p_i(n)\ \mu_i(n)$$

Now we can repeat the argument used to compute $\rho_i(n)$. If we define $z_i(n)$ as the number of customers entering queue i when n-1 customers are already there, during the interval of length T we have

$$z_i(n) = T\ \rho_i(n)$$

Evidently the total number of customers joining station i in time T will be

$$Z_i = \sum_{n=1}^{N} z_i(n) = T \sum_{n=1}^{N} \rho_i(n)$$

Therefore, the frequency $\varphi_i(n)$ with which a customer joins queue i when there are already n-1 customers there will be

$$\varphi_i(n) = \frac{z_i(n)}{Z_i} = \frac{T\ \rho_i(n)}{T \sum_{k=1}^{N} \rho_i(k)}$$

Thus the required probability $Q_i(n)$ is:

$$Q_i(n) = \lim_{T\to\infty} \varphi_i(n) = \lim_{T\to\infty} \frac{T\ \rho_i(n)}{T \sum_{K=1}^{N} \rho_i(k)}$$

$$= \frac{\rho_i(n)}{\sum_{k=1}^{N} \rho_i(k)}$$

and using the definition of $\rho_i(n)$ we can finally write

$$Q_i(n) = \frac{\rho_i(n)}{\sum_{k=1}^{N} \rho_i(k)} = \frac{p_i(n)\ \mu_i(n)}{\sum_{k=1}^{N} p_i(k)\ \mu_i(k)}$$

(Note that for type 3 service centers $\mu_i(n)=n\mu_i$. Also, the quantity

$Q_i(n)$ may be better understood as the distribution of customers at arrival to type 3 centers).

Extensions to Multiple Customer Classes

We will now demonstrate that the above methods can be easily extended to closed queueing networks with multiple custoner classes. We restrict our attention initially to closed networks which do not allow customers to change class membership. In a subsequent section we show that these methods are easily extended once again to handle networks which permit customers to change class. The first order of business is to extend our notation to describe closed queueing networks with multiple job classes.

Network Description (Multiple Customer Class)

The general model consists of a finite number M of service centers and a finite number R of customer classes. Each class has a finite number N_r, $r=1,2,\ldots,R$, of customers. In addition, associated with each job class is a routing probability matrix $p_{ij}(r)$, $i,j=1,2,\ldots,M$ and $r=1,2,\ldots,R$. To be explicit, $p_{ij}(r)$ is the probability that a customer of class r, after completing service at service center i, will next move to service center j. The states of this network are given by the vector $(\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_M)$ where $\underline{n}_i$, $i=1,2,\ldots,M$, is the state of service center i. $\underline{n}_i$ is another vector given by $(n_{i1},n_{i2},\ldots,n_{iR})$

where n_{ir}, $r=1,2,\ldots,R$, is the number of class r customers at service center i. The state $(\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_M)$ is termed feasible if in addition

$$\sum_{i=1}^{M} n_{ir} = N_r \quad \text{and } n_{ir}\geq 0 \text{ for } r=1,2,\ldots,R$$

We will denote a feasible state by $\underline{n} = (\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_M)$ and the set of all feasible states by

$$S(N_1,N_2,\ldots,N_R,M) = \{(\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_M) \mid 0\leq n_{ir}\leq N_r \text{ for } 1\leq i\leq M, \; 1\leq r\leq R \text{ and } \sum_{i=1}^{M} n_{ir} = N_r \text{ for } 1\leq r\leq R\}$$

Note that the transition probabilities are again independent of the state of the network. Each service center is further characterized by service disciplines identical to those previously described.

Again multiple servers will be modelled using load dependent service rates. The general service time distributions for service center types 2 to 4 are required to have rational Laplace transforms.

Joint Probability Distribution (Multiple Customer Class)

Let $p(\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_M)$ be the probability that the system is in state $(\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_M)$ which is assumed to be a feasible state. In [BCMP75, MUN75] it is shown that for closed queueing networks with service centers of types 1,2,3, or 4, the equilibrium state probabilities have the product form:

$$p(\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_M) = \frac{1}{G(N_1,N_2,\ldots,N_R)} \prod_{i=1}^{M} f_i(\underline{n}_i)$$

$$f_i(\underline{n}_i) = \begin{cases} n_i! \prod_{r=1}^{R} \dfrac{1}{n_{ir}!} \left[\dfrac{e_{ir}}{\mu_{ir}} \right]^{n_{ir}} & \text{if service center i is of type 1,2, or 4} \\ \prod_{r=1}^{R} \dfrac{1}{n_{ir}!} \left[\dfrac{e_{ir}}{\mu_{ir}} \right]^{n_{ir}} & \text{if service center i is of type 3} \end{cases} \quad (12)$$

where

$$n_i = \sum_{r=1}^{R} n_{ir}$$

$\dfrac{1}{\mu_{ir}}$ is the mean service time of class r customers at service center i

$$e_{jr} = \sum_{i=1}^{M} e_{ir}\, p_{ij}(r), \qquad j=1,2,\ldots,M \text{ and } r=1,2,\ldots,R$$

and $G(N_1,N_2,\ldots,N_R)$ is the normalization constant obtained by summing all feasible state probabilities in the network model and equating the sum to 1. (Note that the set of numbers e_{jr} are unique up to a multiplicative constant). As before the efficient computation of the normalization constant is essential.

Computation of $G(N_1,N_2,\ldots,N_R)$

The normalization constant is obtained by summing all feasible states of the network and equating the sum to 1, i.e.,

$$G(N_1,N_2,\ldots,N_R) = \sum_{\underline{n}\in S(N_1,N_2,\ldots N_R,M)} \prod_{i=1}^{M} f_i(\underline{n}_i)$$

$$= \sum_{\underline{n}\in S(N_1,N_2,\ldots,N_R,M)} \prod_{i=1}^{M} f_i(n_{i1},n_{i2},\ldots,n_{iR}) \quad (13)$$

As before we define an auxiliary function

$$G_m(n_1,n_2,\ldots,n_R) = \sum_{\underline{n}\in S(n_1,n_2,\ldots,n_R,m)} \prod_{i=1}^{m} f_i(n_{i1},n_{i2},\ldots,n_{iR}) \quad (14)$$

Note that $G(N_1,N_2,\ldots,N_R)$ as defined by (13) is equal to $G_M(N_1,N_2,\ldots,N_R)$. To derive the computational procedure, we start from eq. (14):

$$G_m(n_1,n_2,\ldots,n_R) = \sum_{\underline{n}\in S(n_1,n_2,\ldots,n_R,m)} \prod_{i=1}^{m} f_i(n_{i1},n_{i2},\ldots,n_{iR})$$

$$= \sum_{v_R=0}^{n_R}\cdots\sum_{v_2=0}^{n_2}\sum_{v_1=0}^{n_1} \sum_{\substack{\underline{n}\in S(n_1,n_2,\ldots,n_R,m)\\ n_{m1}=v_1,\ n_{m2}=v_2,\ldots,n_{mR}=v_R}} \prod_{i=1}^{m} f_i(n_{i1},n_{i2},\ldots,n_{iR})$$

$$= \sum_{v_R=0}^{n_R}\cdots\sum_{v_2=0}^{n_2}\sum_{v_1=0}^{n_1} f_m(v_1,v_2,\ldots,v_R) \sum_{\substack{\underline{n}\in S(n_1-v_1,n_2-v_2,\\ \ldots,n_R-v_R,m-1)}} \prod_{i=1}^{m-1} f_i(n_{i1},n_{i2},\ldots n_{iR})$$

$$= \sum_{v_R=0}^{n_R}\cdots\sum_{v_2=0}^{n_2}\sum_{v_1=0}^{n_1} f_m(v_1,v_2,\ldots,v_R)G_{m-1}(n_1-v_1,n_2-v_2,\ldots,n_R-v_R) \quad (15)$$

To compute $G_m(n_1,n_2,\ldots,n_R)$, $n_r=0,1,\ldots,N_r$, $r=1,2,\ldots,R$, we must first compute $G_{m-1}(n_1,n_2,\ldots,n_R)$, $n_r=0,1,\ldots,N_r$, $r=1,2,\ldots,R$. The iterative computation (15) is initialized by setting

$$G_0(n_1,n_2,\ldots,n_R) = \begin{cases} 1 & \text{if } n_1=n_2=\cdots=n_R=0 \\ 0 & \text{else} \end{cases}$$

The iterative relationship defined by (15), together with the initial compute $G_m(n_1,n_2,\ldots,n_R)$ and hence $G_M(N_1,N_2,\ldots,N_R)$.

In light of eq. (14) the discrete case convolution presents itself again. In this case we define the convolution of R dimensional arrays. The structure of each array is as follows: the first component ranges over the values from 0 to N_1, the second component ranges over values from 0 to N_2, ... , and the Rth component ranges over values from 0 to N_R. As an example consider a 2 customer class model (R=2) with N_1=3 and N_2=4. The structure of the arrays is:

$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{ccccc} \quad & \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{array} \right] \end{array}$$

We define a convolution between two R dimensional arrays

$$\vec{A} \qquad \text{and} \qquad \vec{B}$$

of the same component lengths as an array

$$\vec{C}$$

with the same dimension structure as

$$\vec{A} \qquad \text{or} \qquad \vec{B}$$

The elements of the product array are as follows:

$$C(n_1,n_2,\ldots,n_R) = \sum_{v_R=0}^{n_R} \cdots \sum_{v_2=0}^{n_2} \sum_{v_1=0}^{n_1} A(v_1,v_2,\ldots,v_R)B(n_1-v_1,\ldots,n_R-v_R)$$

We represent the convolution as

$$\vec{C} = \vec{A} * \vec{B}$$

We will now rewrite eqns. (14) and (15) in our new notation. First we define an R dimensional array

$$\vec{f}_i$$

with component lengths N_1+1, N_2+1, ... , N_R+1 and whose elements are respectively $f_i(v_1,v_2,\ldots,v_R)$, $v_r=0,1,\ldots,N_r$, $r=1,2,\ldots,R$, and $f_i(\cdot,\cdot,\ldots,\cdot)$ is defined by eq. (12). To illustrate this take the above example. There

$$\vec{f}_i \qquad\qquad \text{becomes}$$

$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{ccccc} f_i(0,0) & f_i(0,1) & f_i(0,2) & f_i(0,3) & f_i(0,4) \\ f_i(1,0) & f_i(1,1) & f_i(1,2) & f_i(1,3) & f_i(1,4) \\ f_i(2,0) & f_i(2,1) & f_i(2,2) & f_i(2,3) & f_i(2,4) \\ f_i(3,0) & f_i(3,1) & f_i(3,2) & f_i(3,3) & f_i(3,4) \end{array}\right] \end{array}$$

Next we define M+1 arrays

$$\vec{G}_0 , \vec{G}_1, \ldots , \vec{G}_M$$

each R dimensional where $\vec{G}_0$ is defined by eq. (16) and

$$\vec{G}_m = \vec{f}_m * \vec{G}_{m-1} \qquad \text{for } m=1,2,\ldots,M \quad \text{(cf. eq. 15)}$$

Notice that the normalization constant

$$G(N_1,N_2,\ldots,N_R) = G_M(N_1,N_2,\ldots,N_R).$$

Marginal Probability (Multiple Customer Class)

Let $p_i(n_1,n_2,\ldots,n_R)$ be the marginal probabiity of finding n_r customers of class r, r=1,2,...,R at station i. Thus,

$$p_i(n_1,n_2,\ldots,n_R)$$

$$= \sum_{\substack{\underline{n}\in S(N_1,N_2,\ldots,N_R,M)\\ \underline{n}_i=S=(n_1,n_2,\ldots,n_R)}} p(\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_{i-1},S,\underline{n}_{i+1},\ldots,\underline{n}_M)$$

$$= \sum_{\substack{\underline{n}\in S(N_1,N_2,\ldots,N_R,M)\\ \underline{n}_i = S}} f_i(S) \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{f_j(\underline{n}_j)}{G(N_1,N_2,\ldots,N_R)}$$

$$= \frac{f_i(S)}{G(N_1,N_2,\ldots,N_R)} \sum_{\substack{\underline{n}\in S(N_1,N_2,\ldots,N_R,M)\\ \underline{n}_i = S}} \prod_{\substack{j=1\\ j\neq i}}^{M} f_j(\underline{n}_j) \qquad (17)$$

Now we redefine our auxiliary function again in order to delete a specific service center from attention, i.e. let

$$G_M^i(N_1-n_1,N_2-n_2,\ldots,N_R-n_R) = \sum_{\substack{\underline{n}\in S(N_1,N_2,\ldots,N_R,M)\\ \underline{n}_i = S}} \prod_{\substack{j=1\\ j\neq i}}^{M} f_j(\underline{n}_j) \qquad (18)$$

Substituting our new auxiliary function definition (18) into (17) we arrive at

$$p_i(n_1,n_2,\ldots,n_R) = \frac{f_i(n_1,n_2,\ldots,n_R)}{G(N_1,N_2,\ldots,N_R)} G_M^i(N_1-n_1,N_2-n_2,\ldots,N_R-n_R) \quad (19)$$

Hence, a procedure to evaluate the auxiliary function (18) is required in order to compute the marginal probability distribution.

Again we are fortunate that a simple recurrence relation for computing (18) exists. The derivation of the algorithm follows from the fact that the marginal probabilities must sum to 1. We have

$$1 = \sum_{n_R=0}^{N_R} \cdots \sum_{n_2=0}^{N_2} \sum_{n_1=0}^{N_1} p_i(n_1,n_2,\ldots,n_R)$$

and

$$G(N_1,N_2,\ldots,N_R) = \sum_{n_R=0}^{N_R} \cdots \sum_{n_2=0}^{N_2} \sum_{n_1=0}^{N_1} \left[f_i(n_1,n_2,\ldots,n_R) \cdot G_M^i(N_1-n_1,N_2-n_2,\ldots,N_R-n_R) \right]$$

Let $\beta(n_1,n_2,\ldots,n_R) = \begin{cases} 0 \text{ if } n_r=0 \text{ for all } r \ (1 \le r \le R) \\ 1 \quad \text{else} \end{cases}$

Then

$$G(N_1,N_2,\ldots,N_R) = f_i(0,0,\ldots,0)\, G_M^i(N_1,N_2,\ldots,N_R) + \sum_{n_R=0}^{N_R} \cdots \sum_{n_2=0}^{N_2} \sum_{n_1=0}^{N_1} \left\{ \beta(n_1,n_2,\ldots,n_R)\, f_i(n_1,n_2,\ldots,n_R) \cdot G_M^i(N_1-n_1,N_2-n_2,\ldots,N_R-n_R) \right] \quad (20)$$

Note $f_i(0,0,\ldots,0_R) = 1$ from eq. (12). So,

$$G_M^i(N_1,N_2,\ldots,N_R) = G(N_1,N_2,\ldots,N_R) - \sum_{n_R=0}^{N_R} \cdots \sum_{n_2=0}^{N_2} \sum_{n_1=0}^{N_1} \left[\beta(n_1,n_2,\ldots,n_R)\, f_i(n_1,n_2,\ldots,n_R) \cdot G_M^i(N_1-n_1,N_2-n_2,\ldots N_R-n_R) \right] \quad (21)$$

Note first that eq. (21) is a recurrence relation in $N_1,N_2,\ldots,N_R$.

The values of $G_M^i(N_1,N_2,\ldots,N_R)$ are computed recursively with the initial condition

$$G_M^i(0,0,\ldots,0_R) = \sum_{\underline{n}\in S(0,0,\ldots,0_R,M)} \prod_{\substack{j=1\\ j\neq i}}^{M} f_j(n_{j1},n_{j2},\ldots,n_{jR})$$

$$= \sum_{\underline{n}\in\{(0,0,\ldots,0_R)\}} \prod_{\substack{j=1\\ j\neq i}}^{M} f_j(0,0,\ldots,0_R) = 1$$

Also note that the β-function appearing in eq. (21) is superfluous if we make the following observation: in order to compute any $G_M(v_1,v_2,\ldots,v_R)$ the only time the β-function is 0 is when $n_1=n_2=\ldots=n_R=0$, i.e. when we also have $G_M(v_1,v_2,\ldots v_R)$ on the right hand side of the equation. Therefore, we may simply initialize $G_M(v_1,v_2,\ldots,v_R)$ to 0. That is <u>not</u> to say that the value of $G_M(v_1,v_2,\ldots,v_R)$ is 0, because we will actually be computing $G_m(v_1,v_2,\ldots,v_R)$ from eq. (20). We introduce the β-function only to exclude from the sum the term we want to compute. The same effect can easily be obtained by initializing that element to 0.

We note in passing that eq. (20) is really an enormous saving in computation over the method of reindexing queues to compute marginal probabiities as in [BUZ73].

<u>Service Center Utilization (Multiple Customer Class)</u>

Let U_{ir} be the utilization of service center i by jobs of class r. Then for any type of service center:

$$U_{ir} = \sum_{n_{iR}=0}^{N_R} \cdots \sum_{n_{ir}=1}^{N_r} \cdots \sum_{n_{i2}=0}^{N_2} \sum_{n_{i1}=0}^{N_1} p_i(n_{i1},n_{i2},\ldots,n_{iR}) \frac{n_{ir}}{n_i} \qquad (22)$$

(More compact formulas for the utilization of a service center will be derived later for special cases).

Throughput (Multiple Customer Class)

Let $TPUT_{ir}$ be the throughput of customers of class r at queue i, i.e. the rate at which customers of class r get serviced and leave queue i. The marginal probabilities that we have computed yield the probability of the configuration $(n_{i1},\ldots,n_{ir},\ldots,n_{iR})$ independent of the ordering of jobs at the i-th station. In order to compute the throughput of class r jobs at station i, we need to know the probability of a class r job being in service. This is the reason the term n_{ir}/n_i appears in the following formulas.

For type 1, 2, or 4 servers:

$$TPUT_{ir} = \sum_{n_{iR}=0}^{N_R} \cdots \sum_{n_{ir}=1}^{N_r} \cdots \sum_{n_{i2}=0}^{N_2} \sum_{n_{i1}=0}^{N_1} p_i(n_{i1},n_{i2},\ldots,n_{iR}) \frac{n_{ir}}{n_i} \mu_{ir}$$

$$= \sum_{n_{iR}=0}^{N_R} \cdots \sum_{n_{ir}=1}^{N_r} \cdots \sum_{n_{i2}=0}^{N_2} \sum_{n_{i1}=0}^{N_1} \left[\frac{f_i(n_{i1},n_{i2},\ldots,n_{iR})}{G_M(N_1,N_2,\ldots,N_R)} \cdot G_M^i(N_1-n_{i1},N_2-n_{i2},\ldots,N_R-n_{iR}) \frac{n_{ir}}{n_i} \mu_{ir} \right]$$

for later reference, observe that:

$$f_i(n_{i1},n_{i2},\ldots,n_{iR})\ \mu_{ir} = n_i! \prod_{j=1}^{R} \left\{ \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}} \right\} \mu_{ir}$$

$$= n_i(n_i-1)!\ \frac{e_{ir}}{\mu_{ir}} \left[\frac{e_{ir}}{\mu_{ir}} \right]^{n_{ir}-1} \frac{1}{n_{ir}(n_{ir}-1)!} \prod_{\substack{j=1\\ j\neq r}}^{R} \left\{ \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}} \right\} \mu_{ir}$$

$$= \frac{n_i}{n_{ir}}\ e_{ir}\ f_i(n_{i1},n_{i2},\ldots,n_{ir}-1,\ldots,n_{iR}) \tag{23}$$

so that

$$TPUT_{ir} = \sum_{n_{iR}=0}^{N_R} \cdots \sum_{n_{ir}=1}^{N_r} \cdots \sum_{n_{i2}=0}^{N_2} \sum_{n_{i1}=0}^{N_1} \left[\frac{e_{ir}}{G_M(N_1,N_2,\ldots,N_R)} \cdot \right.$$

$$f_i(n_{i1},\ldots n_{ir}-1,\ldots,n_{iR}) \cdot$$

$$\left. G_M^i(N_1-n_{i1},N_2-n_{i2},\ldots,N_r-1-n_{ir}+1,\ldots,N_R-n_{iR}) \right]$$

$$= \frac{e_{ir}}{G_M(N_1,N_2,\ldots,N_R)} \sum_{n_{iR}=0}^{N_R} \cdots \sum_{n_{ir}=0}^{N_r-1} \cdots \sum_{n_{i2}=0}^{N_2} \sum_{n_{i1}=0}^{N_1} \left[\right.$$

$$f_i(n_{i1},\ldots,n_{ir},\ldots,n_{iR}) \cdot$$

$$G_M^i(N_1-n_{i1},N_2-n_{i2},\ldots,N_r-n_{ir},\ldots,N_R-N_{iR})\Bigg]$$

$$= \frac{e_{ir}}{G_M(N_1,N_2,\ldots,N_R)} G_M(N_1,N_2,\ldots,N_r-1,\ldots,N_R) \tag{24}$$

For type 3 :

$$TPUT_{ir} = \sum_{n_{iR}=0}^{N_R} \cdots \sum_{n_{ir}=1}^{N_r} \cdots \sum_{n_{i2}=0}^{N_2} \sum_{n_{i1}=0}^{N_1} p_i(n_{i1},n_{i2},\ldots,n_{ir})\, n_{ir}\, \mu_{ir}$$

$$= \sum_{n_{iR}=0}^{N_R} \cdots \sum_{n_{ir}=1}^{N_r} \cdots \sum_{n_{i2}=0}^{N_2} \sum_{n_{i1}=0}^{N_1} \Bigg[\frac{f_i(n_{i1},n_{i2},\ldots,n_{iR})}{G_M(N_1,N_2,\ldots,N_R)}$$

$$G_M^i(N_1-n_{i1},N_2-n_{i2},\ldots,N_R-n_{iR})\; n_{ir}\, \mu_{ir}\Bigg]$$

but again we have that

$$f_i(n_{i1},n_{i2},\ldots,n_{iR})\mu_{ir} = \prod_{j=1}^{R} \left\{ \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}} \right\} \mu_{ir}$$

$$= \frac{1}{n_{ir}(n_{ir}-1)!} \frac{e_{ir}}{\mu_{ir}} \left[\frac{e_{ir}}{\mu_{ir}} \right]^{n_{ir}-1} \prod_{\substack{j=1 \\ j\neq r}}^{R} \left\{ \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}} \right\} \mu_{ir}$$

$$= \frac{1}{n_{ir}} f_i(n_{i1}, n_{i2}, \ldots, n_{ir}-1, \ldots, n_{iR})$$

so that

$$TPUT_{ir} = \sum_{n_{iR}=0}^{N_R} \cdots \sum_{n_{ir}=1}^{N_r} \cdots \sum_{n_{i2}=0}^{N_2} \sum_{n_{i1}=0}^{N_1} \left[\frac{e_{ir}}{G_M(N_1, N_2, \ldots, N_R)} f_i(n_{i1}, \ldots, n_{ir}-1, \ldots, n_{iR}) G_M^i(N_1-n_{i1}, \ldots, N_r-1-n_{ir}+1, \ldots, N_R-n_{iR}) \right]$$

$$= \frac{e_{ir}}{G_M(N_1, N_2, \ldots, N_R)} G_M(N_1, N_2, \ldots, N_r-1, \ldots, N_R) \qquad (25)$$

Notice that the resulting expressions for throughput of a service center have the same form independent of service center type (cf. eqns. (24) and (25)).

Queue Length Distribution at Arrival (Multiple Customer Class)

Let $Q_{ir}(n)$ be the probability that there are n-1 customers already queued (or in service) at station i on the arrival of a job of class r (n=1,...,N where N = total number of customers in closed network). In order to determine $Q_{ir}(n)$ we first find the net rate at which customers of class r enter the i-th queue when there are n-1 customers already present. Let

$$\underline{n}_i^{r-} = (n_{i1}, n_{i2}, \ldots, n_{ir}-1, \ldots, n_{iR})$$

and

$$\underline{n}_j^{r+} = (n_{j1}, n_{j2}, \ldots, n_{jr}+1, \ldots, n_{jR})$$

and define

$$\underline{n}_{ij}^{r} = (\underline{n}_1, \ldots, \underline{n}_i^{r-}, \ldots, \underline{n}_j^{r+}, \ldots, \underline{n}_M)$$

Let $\underline{n} = (\underline{n}_1, \ldots, \underline{n}_i, \ldots \underline{n}_j, \ldots, \underline{n}_M)$ be a feasible state with $i \neq j$. The net rate at which the system transits from state $\underline{n}_{ij}$ to state $\underline{n}$ due to a customer of class r arriving at station i after finishing service at station j is :

For type 1,2 and 4

$$p(\underline{n}_{ij}^{r}) \frac{n_{jr}+1}{n_j+1} \mu_{jr} \, p_{ji}(r) =$$

$$f_1(\underline{n}_1) f_2(\underline{n}_2) \cdots f_i(\underline{n}_i^{r-}) \cdots f_j(\underline{n}_j^{r+}) \cdots f_M(\underline{n}_M) \frac{n_{jr}+1}{n_j+1} \mu_{jr} \frac{p_{ji}(r)}{G(N_1, N_2, \ldots, N_R)}$$

if we now recall equation (23) we can notice that

$$f_j(\underline{n}_j^{r+}) \mu_{jr} = f_j(n_{j1}, n_{j2}, \ldots, n_{jr}+1, \ldots, n_{jR}) \mu_{jr}$$

$$= e_{jr} \frac{n_j+1}{n_{jr}+1} f_j(\underline{n}_j) \qquad (26)$$

so that

$$p(\underline{n}_{ij}^{r}) \frac{n_{jr}+1}{n_j+1} \mu_{jr} \, p_{ji}(r) =$$

$$f_1(\underline{n}_1)f_2(\underline{n}_2)\cdots f_i(\underline{n}_i^{r-})\cdots f_j(\underline{n}_j)\cdots f_M(\underline{n}_M)\ e_{ir}\ p_{ji}(r)\ \frac{1}{G(N_1,N_2,\ldots,N_R)}$$

$$= \frac{1}{G(N_1,N_2,\ldots,N_R)} \left[\prod_{\substack{l=1\\ l\neq i}}^{M} f_l(\underline{n}_l) \right] f_i(\underline{n}_i^{r-})\ e_{jr}\ p_{ji}(r).$$

We can repeat this part of the derivation for type 3 stations as well. The reason we can do this is the same as that given for the derivation of the throughput formulas. Hence, the total net rate at which state $\underline{n}$ is entered due to an arrival of a customer of class r to queue i is

$$e_{ir}(\underline{n}) = \sum_{\substack{j=1\\ j\neq i}}^{M} p(\underline{n}_{ij}^{r})\ \frac{n_{jr}+1}{n_j+1}\ \mu_{jr}p_{ji}(r)\beta(n_{jr}) + p(\underline{n})\ \frac{n_{ir}}{n_i}\ \mu_{ir}p_{ii}(r)\beta(n_{ir})$$

$$= \frac{1}{G(N_1,N_2,\ldots,N_R)} \left\{ \sum_{\substack{j=1\\ j\neq i}}^{M} \left[\prod_{\substack{l=1\\ l\neq i}}^{M} f_l(\underline{n}_l) \right] f_i(\underline{n}_i^{r-})\ e_{jr}\ p_{ji}(r)\ \beta(n_{jr}) + \right.$$

$$\left. \prod_{\substack{l=1\\ l\neq i}}^{M} f_l(\underline{n}_l)\ f_i(\underline{n}_i)\ \mu_{ir}\ p_{ii}(r)\ \frac{n_{ir}}{n_i}\ \beta(n_{ir}) \right\}$$

$$= \frac{1}{G(N_1,N_2,\ldots,N_R)} \sum_{j=1}^{M} \left[\prod_{\substack{l=1\\ l\neq i}}^{M} f_l(\underline{n}_l) \right] f_i(\underline{n}_i^{r-})\ e_{jr}\ p_{ji}(r)\ \beta(n_{jr})$$

$$= \frac{1}{G(N_1,N_2,\ldots,N_R)} \left[\prod_{\substack{l=1 \\ l\neq i}}^{M} f_l(\underline{n}_l) \right] f(\underline{n}_i^{r-}) \; \beta(n_{ir}) \sum_{j=1}^{M} e_{jr} \, p_{ji}(r)$$

$$= \frac{1}{G(N_1,N_2,\ldots,N_R)} \left[\prod_{\substack{l=1 \\ l\neq i}}^{M} f_l(\underline{n}_l) \right] f_i(\underline{n}_i^{r-}) \; \beta(n_{ir}) \; e_{ir}$$

Let us define $\rho'_{ir}(n)$ as the net rate at which customers of class r enter queue i when there are n-1 customers already present there, without imposing any restriction on the population of the other queues (oher than insuring having feasible states). Hence

$$\rho'_{ir}(n) = \sum_{\substack{\underline{n} \,\in\, S(N_1,N_2,\ldots,N_R,M) \\ \sum_{l=1}^{R} n_{il} = n}} \rho_{ir}(\underline{n})$$

$$= \sum_{\substack{\underline{n} \,\in\, S(N_1,N_2,\ldots,N_R,M) \\ \sum_{l=1}^{R} n_{il} = n}} \frac{e_{ir} \; \beta(n_{ir})}{G(N_1,N_2,\ldots,N_R)} \; f_i(\underline{n}_i^{r-}) \prod_{\substack{l=1 \\ l\neq i}}^{M} f_l(\underline{n}_l)$$

$$= \sum_{\substack{S \,\in\, \{(n_1,n_2,\ldots,n_R) \mid \\ \sum_{l=1}^{R} n_l = n \}}} \left\{ \sum_{\substack{\underline{n} \,\in\, S(N_1,N_2,\ldots,N_R,M) \\ \underline{n}_i = S}} \left[\frac{e_{ir} \; \beta(n_{ir})}{G(N_1,N_2,\ldots,N_R)} \right. \right. .$$

$$f_i(\underline{n}_i^{r-}) \prod_{\substack{l=1\\ l\neq i}}^{M} f_l(\underline{n}_l) \Bigg] \Bigg\}$$

Let $S^{r-} = (n_1, n_2, \ldots, n_r - 1, \ldots, n_R)$. Then

$$\varrho'_{ir}(n) = \sum_{S \,\in\, \{(n_1, n_2, \ldots, n_R) \,|\, \sum_{l=1}^{R} n_l = n\}} \Bigg\{ e_{ir}\, \beta(n_r)\, f_i(S^{r-}) \cdot$$

$$\frac{1}{G(N_1, N_2, \ldots, N_R)} \sum_{\substack{\underline{n} \,\in\, S(N_1, N_2, \ldots, N_R, M)\\ \underline{n}_i = S}} \left[\prod_{\substack{l=1\\ l\neq i}}^{M} f_l(\underline{n}_l) \right] \Bigg\}$$

$$= \sum_{S \,\in\, \{(n_1, n_2, \ldots, n_R) \,|\, \sum_{l=1}^{R} n_l = n\}} \Bigg[e_{ir}\, \beta(n_r)\, f_i(S^{r-}) \cdot$$

$$\frac{G_M^i(N_1 - n_1, N_2 - n_2, \ldots, N_R - n_R)}{G(N_1, N_2, \ldots, N_R)} \frac{f_i(S)}{f_i(S)} \Bigg]$$

$$= \sum_{S \in \{(n_1, n_2, \ldots, n_R) \mid \sum_{l=1}^{R} n_l = n\}} \frac{e_{ir}\ \beta(n_r)\ f_i(S^{r-})}{f_i(S)}\ p(S)$$

$$= \sum_{S \in \{(n_1, n_2, \ldots, n_R) \mid \sum_{l=1}^{R} n_l = n\}} \frac{e_{ir}\ \beta(n_r)\ f_i(S^{r-})}{\frac{n}{n_r}\ \frac{e_{ir}}{\mu_{ir}}\ f_i(S^{r-})}\ p(S)$$

$$= \sum_{S \in \{(n_1, n_2, \ldots, n_R) \mid \sum_{l=1}^{R} n_l = n\}} \frac{n_r\ \mu_{ir}\ \beta(n_r)}{n}\ p(S)$$

$$= \sum_{S \in \{(n_1, n_2, \ldots, n_R) \mid \sum_{l=1}^{R} n_l = n\}} \frac{n_r\ \mu_{ir}}{n}\ p(S)$$

Therefore, in analogy with the previous section on queue length distributions we get that the required probability $Q_{ir}(n)$ is

$$Q_{ir}(n) = \frac{\rho'_{ir}(n)}{\sum_{k=1}^{N} \rho'_{ir}(k)}$$

Extension to Load Dependent Service Rates

Let $MU_{ir}(n_{i1},n_{i2},\ldots,n_{iR})$ be the load dependent service rate of station i for customers of class r. Without any loss of generality we can assume a constant element (call it μ_{ir}) can be factored out from the previous expression of MU_{ir} so that we can write

$$MU_{ir}(n_{i1},n_{i2},\ldots,n_{iR}) = \mu_{ir}\, F_{ir}(n_{i1},n_{i2},\ldots,n_{iR}) \qquad (27)$$

The expressions for f_i given in the previous sections result from a solution of the balance equations for a load dependent station. Muntz [MUN73] shows that if we use expression (27) for the load dependent service rate, the solution of the balance equations lead to three useful forms of $F_{ir}(n_{i1},n_{i2},\ldots,n_{iR})$ given by

(I) $$F_{ir}(n_{i1},n_{i2},\ldots,n_{iR}) = g_{ir}(n_{ir})$$

(II) $$F_{ir}(n_{i1},n_{i2},\ldots,n_{iR}) = h_i\left[\sum_{j=1}^{R} n_{ij}\right]$$

(III) $$F_{ir}(n_{i1},n_{i2},\ldots,n_{iR}) = g_{ir}(n_{ir})\; h_i\left[\sum_{j=1}^{R} n_{ij}\right]$$

Notice that (III) is simply the product of the first two expressions.

With every form of $MU_{ir}(n_{i1},n_{i2},\ldots,n_{iR})$ we get a new set of expressions for $f_i(\cdot,\cdot,\ldots,\cdot)$. Specifically we have

(i) $$f_i(n_{i1},n_{i2},\ldots,n_{iR}) = n_i! \prod_{j=1}^{R} \frac{1}{n_{ij}!}\left[\frac{e_{ij}}{\mu_{ij}}\right]^{n_{ij}} \frac{1}{\prod_{k=1}^{n_{ij}} g_{ij}(k)}$$

(ii) $$f_i(n_{i1},n_{i2},\ldots,n_{iR}) = \frac{n_i!}{\prod_{k=1}^{n_i} h_i(k)} \prod_{j=1}^{R} \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}}$$

(iii) $$f_i(n_{i1},n_{i2},\ldots,n_{iR}) = \frac{n_i!}{\prod_{k=1}^{n_i} h_i(k)} \prod_{j=1}^{R} \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}} \frac{1}{\prod_{l=1}^{n_{ij}} g_{ij}(l)}$$

Note: (a) Load dependent service rates can be used only if the service center is of type 1, 2, or 4. (One use of load dependent service rates in these cases is that of modelling multiple server stations.) Load dependent service rates cannot be applied to service stations of type 3 because in this case we would introduce a dependency among the parallel servers of the infinite server station, violating the assumptions that are at the basis of that type of service discipline.

(b) In appendix A we show that for the service discipline FCFS, $g_{ir}(n_{ir})$ must be equal to one for any r and any value of n_{ir}. Hence, the only form of $F_{ir}(n_{i1},n_{i2},\ldots,n_{iR})$ allowed for this kind of service station is

$$F_{ir}(n_{i1},n_{i2},\ldots,n_{iR}) = h_i \left[\sum_{j=1}^{R} n_{ij} \right]$$

Normalization Constant, Marginal Probabilities, and Utilizations

If we study the derivations of these quantities (presented in previous sections), we notice that no derivation required knowledge of the functional form of the f_i's. For this reason we claim that when the f_i's are recomputed according to the new formulas listed above all the results concerning the computation of the normalization constant, the marginal probabilities, and the utilizations remain formally the same.

Throughput and Queue Length Distribution

The derivation of the results concerning the throughputs at the different service stations, as well as the queue length distributions, made an explicit use of the structure of the f_i's.

In order to justify the claim that the results derived in the load independent situation are still formally valid in the load dependent case, we must show that $\mu_{ir}F_{ir}(n_{i1},n_{i2},\ldots,n_{iR})f_i(n_{i1},n_{i2},\ldots,n_{iR})$ can always be expressed in terms of the product

$$e_{ir}f_i(n_{i1},n_{i2},\ldots,(n_{ir}-1),\ldots,n_{iR})$$

and in particular the form of this relation does not depend on the form of the function F_{ir}.

Here we will work out in detail the derivation for one form of f_i and one fundamental form of F_{ir}. Appendix B contains all the other cases derived explicitely.

Suppose we have a service station i of type 2 or 3 and suppose that

$$F_{ir}(n_{i1},n_{i2},\ldots,n_{iR}) = g_{ir}(n_{ir})$$

Then from equation (i) we have:

$$f_{ir}(n_{i1},n_{i2},\ldots,n_{iR}) = n_i! \prod_{j=1}^{R} \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}} \frac{1}{\prod_{k=1}^{n_{ij}} g_{ij}(k)}$$

and

$\mu_{ir}\ g_{ir}(n_{ir})\ f_{ir}(n_{i1},n_{i2},\ldots,n_{iR})$

$$= \mu_{ir}\ g_{ir}(n_{ir})\ n_i! \prod_{j=1}^{R} \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}} \frac{1}{\prod_{k=1}^{n_{ij}} g_{ij}(k)}$$

$$= \mu_{ir}\ g_{ir}(n_{ir})\ n_i\ (n_i-1)! \frac{1}{n_{ir}(n_{ir}-1)!} \frac{e_{ir}}{\mu_{ir}} \left[\frac{e_{ir}}{\mu_{ir}} \right]^{n_{ir}-1} \cdot$$

$$\frac{1}{g_{ir}(n_{ir}) \prod_{l=1}^{n_{ir}-1} g_{ir}(l)} \prod_{\substack{j=1 \\ j\neq r}}^{R} \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}} \frac{1}{\prod_{k=1}^{n_{ij}} g_{ij}(k)}$$

$$= \frac{n_i}{n_{ir}}\ e_{ir}\ f_i(n_{i1},n_{i2},\ldots,n_{ir}-1,\ldots,n_{iR}) \qquad (28)$$

Notice that eq. (28) is the same as eq. (23) used in the derivation of the throughput formula in the load independent case. If we now carefully inspect the derivation of the throughput and queue length distribution results, we can observe that the only place in which we used the explicit structure of the f_i was just during the proof of eq. (23) and (26). This means that if we would repeat those derivations for the load dependent case using eq. (27), we would get the same formal steps and therefore the same formal results.

We can therefore conclude that once the f_i functions are computed according to the new load dependent service rate definitions, all the results (formulas) obtained during the discussion of the load independet case hold also in this new situation!

Extentions to Multiple Customer Classes with Class Switching

Once again we will demonstrate that the above techniques may be genearalized, this time in order to handle closed queueing networks with multiple customer classes where now customers are permitted to change class membership. The following section lists the slight modification to our notation required to describe these networks.

Revised Network Description (Switching between Customer Classes)

Unfortunately with networks that do allow customers to change class, we can no longer guarantee that the number of customers in each class remains constant. However, following [MUN73], we can derive a simple constraint by considering the transition probabilities. For convenience, define a stage of the network to be a pair (i,r) where i is a station index ($1 \le i \le M$) and r is a customer class ($1 \le r \le R$). Movement of customers between the stages of the network is governed by a transition probability matrix $P=[p_{ir\ js}]$, where the $p_{ir\ js}$ are defined as above.

We are interested in partitioning the set of customer classes $\{1,2,\ldots,R\}$ into disjoint subsets such that a customer which initially belongs to one class of a subset can possibly find himself in any other class of the same subset. If no switches between classes are permitted, then each subset contains exactly one class. Since we are assuming that customer classes in different subsets cannot communicate, then the total number of customers in any subset must remain constant. In order to insure that the network model is solvable, we make one additional restriction: if class r and class s belong to the same subset, then any stage (j,s) which class s can visit must be reachable from any stage (i,r) which class r can visit.

More formally, define

$$E_r = \{\, s \mid \text{stage } (j,s) \text{ can be reached in zero or more transitions from stage } (i,r) \text{ for any } i,j=1,2,\ldots,M\}, \text{ for } 1\le r,s\le R,$$

$$= \{\, s \mid P^{n}_{(i,r);(j,s)} \ne 0,\ n\ge 0,\ i,j=1,2,\ldots,M\},\ \text{for } 1\le r,s\le R.$$

This actually defines R sets, one for each customer class. From these R sets eliminate any duplicate sets. Label the remaining U sets EC_1, $EC_2,\ldots,EC_U$. Let

$$N_s = \sum_{r\in EC_s} N'_r, \qquad s=1,2,\ldots,U.$$

Then N_s is the constant number of customers in the set EC_s.

Now we can explicitly define the set of all feasible states to be

$$S(N_1,N_2,\ldots,N_U,M) = \{\, (\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_M) \mid \underline{n}_i = (n_{i1},n_{i2},\ldots,n_{iR}),\ n_{ir}\ge 0,\ i=1,2,\ldots,M\ ,\ r=1,2,\ldots,R, \text{ and } \sum_{\substack{r\in EC_s\\ 1\le i\le M}} n_{ir} = N_s,\ s=1,2,\ldots,U\}.$$

The total number of feasible states is

$$\prod_{s=1}^{U} \binom{M \cdot C_S + N_S - 1}{M \cdot C_S - 1}$$

where C_S is the cardinality of EC_S.

At this point an example will help clarify these ideas. Consider a closed queueing network with three customer classes (R=3) and two servers (M=2). Let the total number of customers in the network be 10 with these customers being initially distributed among the customer classes as follows: $N'_1=2$, $N'_2=3$, and $N'_3=5$. Let

$$P = \begin{array}{c} \\ (1,1) \\ (1,2) \\ (1,3) \\ (2,1) \\ (2,2) \\ (2,3) \end{array} \begin{array}{c} \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (2,1) & (2,2) & (2,3) \end{array} \\ \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 3/4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \end{array}$$

be the transition probability matrix. Then $E_1=\{1,3\}$, $E_2=\{2\}$, and $E_3=\{1,3\}$. Eliminating duplicate sets, we have $EC_1=\{1,3\}$ and $EC_2=\{2\}$. Furthermore, $N_1=N'_1+N'_3=7$, $N_2=3$, $C_1=2$, $C_2=1$, and there are 480 feasible states in this model.

<u>Computation of $G(N_1,N_2,\ldots,N_U)$</u>

The first step is to note as above that the equilibrium state probabilities must sum to 1. Hence,

$$G(N_1,N_2,\ldots,N_U) = \sum_{\underline{n} \in S(N_1,N_2,\ldots,N_U,M)} \prod_{i=1}^{M} f_i(\underline{n}_i)$$

$$= \sum_{\underline{n} \in S(N_1,N_2,\ldots,N_U,M)} \prod_{i=1}^{M} f_i(n_{i1},n_{i2},\ldots,n_{iR}) \quad (29)$$

In order to derive the algorithm it is necessary to define an auxiliary function similiar to [BUZ73]. Let

$$G_m(n_1,n_2,\ldots,n_U) = \sum_{\underline{n} \in S(n_1,n_2,\ldots,n_U,m)} \prod_{i=1}^{m} f_i(\underline{n}_i) \quad (30)$$

Note that $G(N_1,N_2,\ldots,N_U)$ as defined by (29) is equal to $G_M(N_1,N_2,\ldots,N_U)$. To derive the computational procedure, we start from eq. (30):

$$G_m(n_1,n_2,\ldots,n_U) = \sum_{\underline{n} \in S(n_1,n_2,\ldots,n_U,m)} \prod_{i=1}^{m} f_i(n_{i1},n_{i2},\ldots,n_{iR})$$

$$= \sum_{v_U=0}^{n_U} \cdots \sum_{v_2=0}^{n_2} \sum_{v_1=0}^{n_1} \left[\sum_{\substack{\underline{n} \in S(n_1,n_2,\ldots,n_U,m) \\ \sum_{k \in EC_l} n_{mk} = v_l,\ l=1,2,\ldots,U}} \prod_{i=1}^{m} f_i(n_{i1},n_{i2},\ldots,n_{iR}) \right]$$

$$= \sum_{v_U=0}^{n_U} \cdots \sum_{v_2=0}^{n_2} \sum_{v_1=0}^{n_1} \left[\sum_{\substack{S_m \in \{(n_{m1},n_{m2},\ldots,n_{mR}) \mid \\ \sum_{k \in EC_l} n_{mk} = v_l,\ l=1,2,\ldots,U\}}} f_m(S_m) \cdot \sum_{\substack{\underline{n} \in S(n_1,n_2,\ldots,n_U,m) \\ n_m = S_m}} \prod_{i=1}^{m-1} f_i(n_{i1},n_{i2},\ldots,n_{iR}) \right]$$

$$= \sum_{v_U=0}^{n_U} \cdots \sum_{v_2=0}^{n_2} \sum_{v_1=0}^{n_1} \left[\sum_{\substack{S_m \in \{(n_{m1}, n_{m2}, \ldots, n_{mR}) \mid \\ \sum_{k \in EC_l} n_{mk} = v_l,\ l=1,2,\ldots,U\}}} f_m(S_m) \cdot \sum_{\underline{n} \in S(n_1-v_1, n_2-v_2, \ldots, n_U-v_U, m-1)} \prod_{i=1}^{m-1} f_i(n_{i1}, n_{i2}, \ldots, n_{iR}) \right]$$

$$= \sum_{v_U=0}^{n_U} \cdots \sum_{v_2=0}^{n_2} \sum_{v_1=0}^{n_1} \left[\sum_{\substack{S_m \in \{(n_{m1}, n_{m2}, \ldots, n_{mR}) \mid \\ \sum_{k \in EC_l} n_{mk} = v_l,\ l=1,2,\ldots,U\}}} f_m(n_{m1}, n_{m2}, \ldots, n_{mR}) \cdot G_{m-1}(n_1-v_1, n_2-v_2, \ldots, n_U-v_U) \right] \tag{31}$$

To compute $G_m(n_1, n_2, \ldots, n_U)$, $n_i=0,1,\ldots,N_i$, $i=1,2,\ldots,U$, we must first compute $G_{m-1}(n_1, n_2, \ldots, n_U)$, $n_i=0,1,\ldots,N_i$, $i=1,2,\ldots,U$. The iterative computation (31) is initialized by setting

$$G_0(n_1, n_2, \ldots, n_U) = \begin{cases} 1 & \text{if } n_1=n_2=\cdots=n_U=0 \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

The iterative relationship defined by (31) and the initial conditions specified by (32) completely determine the algorithm to compute $G_m(n_1, n_2, \ldots, n_U)$ and hence the normalization constant $G_M(N_1, N_2, \ldots, N_U)$.

Again we note in passing that eq. (31) is similar to a discrete case convolution. See [CHW75, REK75] for further details.

Marginal Probabilities

Let $p_i(n_1,n_2,\ldots,n_r,\ldots,n_R)$ be the marginal probability that there are n_r customers of class r ($r=1,2,\ldots,R$) at service center i. Then

$$p_i(n_1,n_2,\ldots,n_R) = \sum_{\substack{\underline{n}\in S(N_1,N_2,\ldots,N_U,M)\\ \underline{n}_i=S=(n_1,n_2,\ldots,n_R)}} p(\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_{i-1},S,\underline{n}_{i+1},\ldots,\underline{n}_M)$$

$$= \sum_{\substack{\underline{n}\in S(N_1,N_2,\ldots,N_U,M)\\ \underline{n}_i=S=(n_1,n_2,\ldots,n_R)}} \frac{1}{G(N_1,N_2,\ldots,N_U)} f_i(S) \prod_{\substack{j=1\\ j\neq i}}^{M} f_j(\underline{n}_j)$$

$$= \frac{f_i(n_1,n_2,\ldots,n_R)}{G(N_1,N_2,\ldots,N_U)} \sum_{\substack{\underline{n}\in S(N_1,N_2,\ldots,N_U,M)\\ \underline{n}_i=S=(n_1,n_2,\ldots,n_R)}} \prod_{\substack{j=1\\ j\neq i}}^{M} f_j(n_{j1},n_{j2},\ldots,n_{jR}) \quad (33)$$

Define another auxiliary function related to our previous auxiliary function but which permits us to delete a specific service center from attention, i.e. let

$$G_M^i(N_1-v_1,N_2-v_2,\ldots,N_U-v_U) = \sum_{\underline{n}\in S(N_1,N_2,\ldots,N_U,M)} \prod_{\substack{j=1\\ j\neq i}}^{M} f_j(n_{j1},n_{j2},\ldots,n_{jR}) \quad (34)$$

$$v_l = \sum_{k\in EC_l} n_{ik}\ ,\quad l=1,2,\ldots,U$$

Now substituting our new auxiliary function definition (34) into (33) we arrive at

$$p_i(n_1,n_2,\ldots,n_R) = \frac{f_i(n_1,n_2,\ldots,n_R)}{G(N_1,N_2,\ldots,N_U)} G_M^i(N_1-v_1,N_2-v_2,\ldots,N_U-v_U)$$

where $v_l = \sum_{k\in EC_l} n_k\ ,\quad l=1,2,\ldots,U$

Hence, in order to compute the marginal probabilities, an efficient procedure to evaluate the auxiliary function (34) is required.

Once again, a simple recurrence relation for computing (34) exists. The relation follows from the fact that the marginal probabilities must sum to 1. Therefore,

$$1 = \sum_{v_U=0}^{N_U} \cdots \sum_{v_2=0}^{N_2} \sum_{V_1=0}^{N_1} \sum_{\substack{S\in\{(n_{i1},n_{i2},\ldots,n_{iR}) \mid \\ \sum_{k\in EC_l} n_{ik} = v_l \;,\; l=1,2,\ldots,U\}}} p_i(S)$$

$$= \sum_{v_U=0}^{N_U} \cdots \sum_{v_2=0}^{N_2} \sum_{V_1=0}^{N_1} \left[\sum_{\substack{S\in\{(n_{i1},n_{i2},\ldots,n_{iR}) \mid \\ \sum_{k\in EC_l} n_{ik} = v_l \;,\; l=1,2,\ldots,U\}}} \frac{f_i(S)}{G_M(N_1,N_2,\ldots,N_U)} \cdot \sum_{\substack{\underline{n}\in S(N_1,N_2,\ldots,N_U,M) \\ \underline{n}_i=S}} \prod_{\substack{j=1 \\ j\neq i}}^{M} f_j(\underline{n}_j) \right]$$

from which

$$G_M(N_1,N_2,\ldots,N_U) = \sum_{v_U=0}^{N_U} \cdots \sum_{v_2=0}^{N_2} \sum_{V_1=0}^{N_1} \left[\sum_{\substack{S\in\{(n_{i1},n_{i2},\ldots,n_{iR}) \mid \\ \sum_{k\in EC_l} n_{ik} = v_l \;,\; l=1,2,\ldots,U\}}} f_i(S) \cdot \sum_{\substack{\underline{n}\in S(N_1,N_2,\ldots,N_U,M) \\ \underline{n}_i=S}} \prod_{\substack{j=1 \\ j\neq i}}^{M} f_j(\underline{n}_j) \right]$$

$$= \sum_{v_U=0}^{N_U} \cdots \sum_{v_2=0}^{N_2} \sum_{v_1=0}^{N_1} \left[\sum_{\substack{S\epsilon\{(n_{i1},n_{i2},\ldots,n_{iR}) \mid \\ \sum_{k\epsilon EC_l} n_{ik} = v_l \;,\; l=1,2,\ldots,U\}}} f_i(S) \cdot G_M^{i}(N_1-v_1,N_2-v_2,\ldots,N_U-v_U) \right]$$

Let

$$\beta(v_1,v_2,\ldots,v_U) = \begin{cases} 0 & \text{if } v_i=0 \quad i=1,2,\ldots,U \\ 1 & \text{otherwise} \end{cases}$$

Then

$$G_M(N_1,N_2,\ldots,N_U) = f_i(0,0,\ldots,0)\; G_M^{i}(N_1,N_2,\ldots,N_U) +$$
$$\sum_{v_U=0}^{N_U} \cdots \sum_{v_2=0}^{N_2} \sum_{v_1=0}^{N_1} \left[\sum_{\substack{S\epsilon\{(n_{i1},n_{i2},\ldots,n_{iR}) \mid \\ \sum_{k\epsilon EC_l} n_{ik} = v_l \;,\; l=1,2,\ldots,U\}}} \beta(v_1,v_2,\ldots,v_U) f_i(S) G_M^{i}(N_1-v_1,\ldots,N_U-v_U) \right]$$

Since $f_i(0,0,\ldots,0) = 1$ we have

$$G_M^{i}(N_1,N_2,\ldots,N_U) = G_M(N_1,N_2,\ldots,N_U) - \tag{35}$$
$$\sum_{v_U=0}^{N_U} \cdots \sum_{v_2=0}^{N_2} \sum_{v_1=0}^{N_1} \left[\sum_{\substack{S\epsilon\{(n_{i1},n_{i2},\ldots,n_{iR}) \mid \\ \sum_{k\epsilon EC_l} n_{ik} = v_l \;,\; l=1,2,\ldots,U\}}} \beta(v_1,v_2,\ldots,v_U) f_i(S) G_M^{i}(N_1-v_1,\ldots,N_U-v_U) \right]$$

Note first that eq. (35) is a recurrence relation in N_1, $N_2,\ldots,N_U$. The values of $G_M(N_1,N_2,\ldots,N_U)$ are computed recursively with the initial conditions

$$G_M^i(0,0,\ldots,0_U) = 1$$

Also note that the β-function appearing in eq. (35) is superfluous if we make the following observation: in order to compute any $G_M(N_1,N_2,\ldots,N_U)$ the only time the β-function is 0 is when $v_1=v_2=\cdots=v_U=0$, i.e., in the term with $G_M(N_1,N_2,\ldots,N_U)$ on the right hand side. Therefore, an algorithm to compute (35) can take advantage of this fact by initializing $G_M(N_1,N_2,\ldots,N_U)$ to 0 and then computing $G_M(N_1,N_2,\ldots,N_U)$ by eq. (35) with the β-function omitted.

Note carefully that this method of computing marginal probabilities results in a considerable savings in computation over the method of reindexing of queues as employed by [BUZ73, MUW74, CHW75].

Service Center Utilization

Let U_{ir} be the utilization of service center i by customers of class r. Then for any type of service center

$$U_{ir} = \sum_{v_U=0}^{N_U} \cdots \sum_{v_q=0}^{N_q} \cdots \sum_{v_1=0}^{N_1} \sum_{\substack{S\epsilon\{(n_{i1},n_{i2},\ldots,n_{iR})\,|\\ \sum_{k\epsilon EC_l} n_{ik} = v_l\ ,\ l=1,2,\ldots U\ \&\ n_{ir}>0\}}} p_i(n_{i1},n_{i2},\ldots,n_{iR})\ \frac{n_{ir}}{n_i}$$

The condition $n_{ir}>0$ may be removed, yielding

$$U_{ir} = \sum_{v_U=0}^{N_U} \cdots \sum_{v_q=0}^{N_q} \cdots \sum_{v_1=0}^{N_1} \sum_{\substack{S\epsilon\{(n_{i1},n_{i2},\ldots,n_{iR})\,|\\ \sum_{k\epsilon EC_l} n_{ik} = v_l\ ,\ l=1,2,\ldots,U\}}} p_i(n_{i1},n_{i2},\ldots,n_{iR})\ \frac{n_{ir}}{n_i}$$

Throughput and Mean Response Time

Let $TPUT_{ir}$ be the throughput of customers of class r at server i, i.e. the rate at which customers of class r get serviced and leave server i. The marginal probabilities that we have computed yield the probability of the configuration $(n_{i1}, n_{i2}, \ldots, n_{ir}, \ldots, n_{iR})$ independent of the ordering of customers at the i-th station. In order to compute the throughput of class r customers at server i, we need to know the probability of a class r customer being in service. This is the reason the term n_{ir}/n_i appears in the following formulas

recall $n_i = \sum_{r=1}^{R} n_{ir}$ and let q be such that $r \in EC_q$

For type 1, 2, and 4 servers:

$$TPUT_{ir} = \sum_{v_U=0}^{N_U} \cdots \sum_{v_q=1}^{N_q} \cdots \sum_{v_1=0}^{N_1} \mu_{ir} \sum_{\substack{S \in \{(n_{i1}, n_{i2}, \ldots, n_{iR}) \mid \\ \sum_{k \in EC_l} n_{ik} = v_l \ , l=1,2,\cdots,U \ \& \ n_{ir}>0\}}} p_i(n_{i1}, n_{i2}, \ldots, n_{iR}) \frac{n_{ir}}{n_i}$$

observe that

$$\mu_{ir} \frac{n_{ir}}{n_i} f_i(n_{i1}, n_{i2}, \ldots, n_{iR}) = e_{ir} \ f_i(n_{i1}, n_{i2}, \ldots, n_{ir}-1, \ldots, n_{iR})$$

thus

$$TPUT_{ir} = \sum_{v_U=0}^{N_U} \cdots \sum_{v_q=1}^{N_q} \cdots \sum_{v_1=0}^{N_1} \left[\sum_{\substack{S \in \{(n_{i1}, n_{i2}, \ldots, n_{iR}) \mid \\ \sum_{k \in EC_l} n_{ik} = v_l \ , l=1,2,\cdots,U \ \& \ n_{ir}>0\}}} \mu_{ir} \frac{n_{ir}}{n_i} f_i(S) \right.$$

$$\left. \frac{G_M^i(N_1-v_1, N_2-v_2, \ldots, N_U-v_U)}{G_M(N_1, N_2, \ldots, N_U)} \right]$$

$$= \sum_{v_U=0}^{N_U} \cdots \sum_{v_q=1}^{N_q} \cdots \sum_{v_1=0}^{N_1} \left[\sum_{\substack{S\epsilon\{(n_{i1}, n_{i2}, \ldots n_{iR}) \mid \\ \sum_{k\epsilon EC_l} n_{ik} = v_l \;, l=1,2,\cdots,U \;\&\; n_{ir}>0\}}} e_{ir}\, f_i(n_{i1}, n_{i2}, \ldots, n_{ir}-1, \ldots, n_{iR}) \right.$$

$$\left. \frac{G_M^i(N_1-v_1, N_2-v_2, \ldots, N_U-v_U)}{G_M(N_1, N_2, \ldots, N_U)} \right]$$

$$= \sum_{v_U=0}^{N_U} \cdots \sum_{v_q=o}^{N_q-1} \cdots \sum_{v_1=0}^{N_1} \left[\sum_{\substack{S\epsilon\{(n_{i1}, n_{i2}, \ldots, n_{iR}) \mid \\ \sum_{k\epsilon EC_l} n_{ik} = v_l \;, l1,2,\cdots,U\}}} e_{ir}\, f_i(n_{i1}, n_{i2}, \ldots, n_{ir}, \ldots, n_{iR}) \right.$$

$$\left. \frac{G_M^i(N_1-v_1, N_2-v_2, \ldots, N_q-1-v_q, \ldots, N_U-v_U)}{G_M(N_1, N_2, \ldots, N_U)} \right]$$

$$= e_{ir} \frac{G_M(N_1, N_2, \ldots, N_q-1, \ldots, N_U)}{G_M(N_1, N_2, \ldots, N_U)}$$

For type 3 servers:

$$TPUT_{ir} = \sum_{v_U=0}^{N_U} \cdots \sum_{v_q=1}^{N_q} \cdots \sum_{v_1=0}^{N_1} \mu_{ir} n_{ir} \sum_{\substack{S \in \{(n_{i1}, n_{i2}, \ldots, n_{iR}) \mid \\ \sum_{k \in EC_l} n_{ik} = v_l \ , l=1,2,\cdots,U \ \& \ n_{ir}>0\}}} p_i(n_{i1}, n_{i2}, \ldots, n_{iR})$$

but since

$$\mu_{ir} \ n_{ir} \ f_i(n_{i1}, n_{i2}, \ldots, n_{iR}) = e_{ir} \ f_i(n_{i1}, n_{i2}, \ldots, n_{ir}-1, \ldots, n_{iR})$$

we have

$$TPUT_{ir} = \sum_{v_U=0}^{N_U} \cdots \sum_{v_q=1}^{N_q} \cdots \sum_{v_1=0}^{N_1} \left[\sum_{\substack{S \in \{(n_{i1}, n_{i2}, \ldots, n_{iR}) \mid \\ \sum_{k \in EC_l} n_{ik} = v_l \ , l1,2,\cdots,U \ \& \ n_{ir}>0\}}} \mu_{ir} \ n_{ir} \ f_i(S) \right.$$

$$\left. \frac{G_M^i(N_1-v_1, N_2-v_2, \ldots, N_U-v_U)}{G_M(N_1, N_2, \ldots, N_U)} \right]$$

$$= e_{ir} \frac{G_M(N_1, N_2, \ldots, N_q-1, \ldots, N_U)}{G_M(N_1, N_2, \ldots, N_U)} \qquad (36)$$

Hence, we have the same expression for throughput regardless of the service center type.

Let W_{ir} be the mean response (waiting) time of class r customers at service center i, i.e. the average time a customer spends at service center i (queueing time plus service time). This can be determined for systems in equilibrium using Little's formula [LIT61]:

$$W_{ir} = \bar{n}_{ir} \ / \ \lambda_{ir}$$

where

$$\bar{n}_{ir} \text{ and } \lambda_{ir}$$

are the average number and average arrival rate of class r customers to server i, respectively. Since the throughput of a service center is equal to the arrival rate to that service center if the system is in equilibrium, then

$$\lambda_{ir} = TPUT_{ir} \quad ,i=1,2,\cdots,M \text{ and } r=1,2,\cdots,R$$

Also, $\bar{n}_{ir}$ may be computed as

$$\bar{n}_{ir} = \sum_{v_U=0}^{N_U} \cdots \sum_{v_q=1}^{N_q} \cdots \sum_{v_1=0}^{N_1} \sum_{\substack{S\epsilon\{(n_{i1},n_{i2},\ldots,n_{iR}) \mid \\ \sum_{k\epsilon EC_l} n_{ik} = v_l \;,l=1,2,\cdots,U\}}} p(S)\; n_{ir}$$

Therefore

$$W_{ir} = \frac{\bar{n}_{ir}}{TPUT_{ir}}$$

<u>More on Service Center Utilization</u>

The formulas for the utilization of a service center that have been presented in a previous section of this paper are derived in a straightforward manner from the formal definition of utilization. These formulas hold for the most general cases but are more complex than often times necessary.

For example, if the service center has a load independent service rate, it is possible to show that the utilization can be written in the following form:

$$U_{ir} = \frac{TPUT_{ir}}{\mu_{ir}} = \frac{\lambda_{ir}}{\mu_{ir}}$$

which is valid also in the case in which switching between different customer classes is permitted.

In the load dependent case no simplifications of the general utilization formula seem possible.

The modelling of service stations with multiple servers, via the introduction of a load dependency function in a service center with a single server, requires a careful interpretation of our definition of utilization. Our utilization formula implies that a service center (even with multiple servers) is fully utilized when one or more customers are present at the station. The desired value for the utilization of the service center can be computed using knowledge of the marginal probabilities.

Also, if we can assume that a customer joining an empty station with multiple identical servers chooses a server randomly, it is possible to show in such a case that

$$U_{ir} = \frac{\lambda_{ir}}{a_i \mu_{ir}}$$

where a_i is the number of identical servers at that station.

APPENDIX A

Let $X=(x_1,x_2,\ldots,x_n)$ be a vector representing the configuration of the queue of a FCFS service station, where the value of each component x_i of X represents the class of the customer in the i-th position of the queue. With this notation x_1 represents the job being processed by the service station and therefore corresponds to the head of the queue. Let $X_r=(r,x_1,x_2,\ldots,x_n)$ be a vector representing the configuration of the queue of the same service station when a customer of class r is at the head of the queue (being served).

The property that arrival Poisson processes imply departure Poisson processes [MUN73,MUN75], allows us to write down the following equations

$$\frac{p(r,x_1,x_2,\ldots,x_n)\ \mu_r}{p(x_1,x_2,\ldots,x_n)} = \lambda_r$$

that is

$$p(r,x_1,x_2,\ldots,x_n) = \frac{\lambda_r}{\mu_r}\ p(x_1,x_2,\ldots,x_n)$$

If we use the above equation to recursively compute $p(x_1,x_2\ldots,x_n)$ and if we notice that among the n customers in the queue, n_i is the number of customers of class i, i=1,2,...,R (where R is the total number of classes considered in the model), then we can write

$$p(x_1,x_2,\ldots,x_n) = p(0)\prod_{j=1}^{R}\left[\frac{\lambda_j}{\mu_j}\right]^{n_j}$$

In general if we consider all the combinations with repetition of n customers of different classes in the queue, we will get

$$p(n_1,n_2,\ldots,n_R) = \frac{n!}{n_1!\ n_2!\ \cdots\ n_R!}\ p(0) \prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j}\right]^{n_j}$$

$$= p(0)\ n! \prod_{j=1}^{R} \frac{1}{n_j!} \left[\frac{\lambda_j}{\mu_j}\right]^{n_j}$$

where

$$n = \sum_{k=1}^{R} n_k$$

We can now write the balance equation (Flow in = Flow out) for this service station

$$\begin{aligned} p(x_1,x_2,\ldots,x_n) \sum_{r=1}^{R} \lambda_r &+ p(x_1,x_2,\ldots,x_n)\ \mu_{x_1} \\ = \sum_{r=1}^{R} p(r,x_1,x_2,\ldots,x_n)\mu_r &+ p(x_1,x_2,\ldots,x_{n-1})\ \lambda_{x_n} \end{aligned} \tag{A1}$$

Substituting the expression of $p(\cdot,\cdot,\ldots,\cdot)$ we get

$$p(0) \prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j}\right]^{n_j} \left\{ \sum_{r=1}^{R} \lambda_r + \mu_{x_1} \right\}$$

$$= \sum_{r=1}^{R} \frac{\lambda_r}{\mu_r} \mu_r\ p(0) \prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j}\right]^{n_j} + p(0) \prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j}\right]^{n_j} \frac{\mu_{x_n}}{\lambda_{x_n}}\ \lambda_{x_n}$$

and therefore

$$\prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j} \right]^{n_j} \mu_{x_1} = \prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j} \right]^{n_j} \mu_{x_n}$$

so that we can conclude that

$$\mu_{x_1} = \mu_{x_n}$$

for every n, i.e. the average service rate of customers of different classes must be the same. Thus we get Poisson departures from a FCFS system if and only if the service times for all classes of customers are exponentially distributed with the same mean.

Load Dependent Case

We can now repeat the previous derivation for the case in which the service rate MU_r of customers of class r has the following form

$$MU_r = \mu_r \, F_r(n_1, n_2, \ldots, n_R)$$

The Poisson condition on the arrival and departure processes of the service station can now be written as

$$\frac{p(r, x_1, x_2, \ldots, x_n) \;\mu_r \; F_r(n_1, n_2, \ldots, n_r+1, \ldots, n_R)}{p(x_1, x_2, \ldots, x_n)} = \lambda_r$$

that is

$$p(r, x_1, x_2, \ldots, x_n) = \frac{\lambda_r}{\mu_r \; F_r(n_1, n_2, \ldots, n_r+1, \ldots, n_R)} \; p(x_1, x_2, \ldots, x_n)$$

In particular if

$$F_r(n_1, n_2, \ldots, n_r+1, \ldots, n_R) = g_r(n_r+1)$$

we get

$$p(r, x_1, x_2, \ldots, x_n) = \frac{\lambda_r}{\mu_r \; g_r(n_r+1)} \; p(x_1, x_2, \ldots, x_n)$$

and therefore

$$p(x_1, x_2, \ldots, x_n) = p(0) \prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j} \right]^{n_j} \frac{1}{\prod_{k=1}^{n_j} g_j(k)}$$

where $n = \sum_{k=1}^{R} n_k$

If we now substitute this expression in the balance equation (A1) we will get

$$p(0) \prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j} \right]^{n_j} \frac{1}{\prod_{k=1}^{n_j} g_j(k)} \left\{ \sum_{r=1}^{R} \lambda_r + \mu_{x_i} \, g_{x_i}(n_{x_i}) \right\}$$

$$= \sum_{r=1}^{R} \left\{ p(0) \frac{\lambda_r}{\mu_r} \frac{1}{g_r(n_r+1)} \prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j} \right]^{n_j} \frac{1}{\prod_{k=1}^{n_j} g_j(k)} \mu_r \; g_r(n_r+1) \right\} +$$

$$p(0) \prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j} \right]^{n_j} \frac{1}{\prod_{k=1}^{n_j} g_j(k)} \left[\frac{\lambda_{x_n}}{\mu_{x_n}} \right]^{n_{x_n}-1} \frac{1}{\prod_{k=1}^{n_{x_n}-1} g_{x_n}(k)} \lambda_{x_n} \frac{\lambda_{x_n}}{\mu_{x_n}} \frac{\mu_{x_n}}{\lambda_{x_n}} \frac{g_{x_n}(n_{x_n})}{g_{x_n}(n_{x_n})}$$

from which follows

$$\mu_{x_1} g_{x_1}(n_{x_1}) = \mu_{x_n} g_{x_n}(n_{x_n})$$

but since n, n_{x_n} and the class x_n are all arbitrary values, the only way to have the previous relation satisfied is that of imposing

$$g_i(n_i) = 1 \quad i=1,2,\ldots,R$$

If instead we chose the following load dependency function

$$F_r(n_1,n_2,\ldots,n_R) = h\left[\sum_{k=1}^{R} n_k\right]$$

then the Poisson arrival-departure condition gives

$$p(r,x_1,x_2,\ldots,x_n) = \frac{\lambda_r}{\mu_r\, h\left[1+\sum_{k=1}^{R} n_k\right]}\, p(x_1,x_2,\ldots,x_n)$$

and if $n = \sum_{k=1}^{R} n_k$ we get

$$p(x_1,x_2,\ldots,x_n) = \frac{1}{\prod_{k=1}^{n} h(k)}\, p(0) \prod_{j=1}^{R}\left[\frac{\lambda_j}{\mu_j}\right]^{n_j}$$

Now the balance equation (A1) becomes

$$\frac{1}{\prod_{k=1}^{n} h(k)}\, p(0) \prod_{j=1}^{R}\left[\frac{\lambda_j}{\mu_j}\right]^{n_j}\left\{\sum_{r=1}^{R}\lambda_r + \mu_{x_1} h(n)\right\}$$

$$= \sum_{r=1}^{R} \left\{ \frac{1}{\prod_{k=1}^{n+1} h(k)} \; p(0) \; \frac{\lambda_r}{\mu_r} \prod_{j=1}^{R} \left[\frac{\lambda_j}{\mu_j} \right]^{n_j} \mu_r \; h(n+1) \right\} \div$$

$$\frac{1}{\prod_{k=1}^{n-1} h(k)} \; p(0) \prod_{\substack{j=1 \\ j \neq x_n}}^{R} \left[\frac{\lambda_j}{\mu_j} \right]^{n_j} \left[\frac{\lambda_{x_n}}{\mu_{x_n}} \right]^{n_{x_n} - 1} \frac{h(n)}{h(n)} \; \frac{\lambda_{x_n}}{\mu_{x_n}} \; \frac{\mu_{x_n}}{\lambda_{x_n}} \; \lambda_{x_n}$$

From which we can conclude

$$\mu_{x_1} = \mu_{x_n} \qquad \text{for every } n$$

and thus the form of the load dependency function, in this case, can be arbitrary.

APPENDIX B

Consider the following form of the load dependency function

$$F_{ir}(n_{i1}, n_{i2}, \ldots, n_{iR}) = h_i \left[\sum_{j=1}^{R} n_{ij} \right]$$

then for service centers of type 1, 2, and 4 we get

$$\mu_{ir} \; h_i(n_i) \; f_i(n_{i1}, n_{i2}, \ldots, n_{iR})$$

$$= \mu_{ir} \; h_i(n_i) \; \frac{n_i!}{\prod_{k=1}^{n_i} h_i(k)} \prod_{j=1}^{R} \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}}$$

$$= \mu_{ir}\ h_i(n_i)\ \frac{n_i\ (n_i-1)!}{h_i(n_i)\ \prod_{k=1}^{n_i-1} h_i(k)}\ \frac{1}{n_{ir}\ (n_{ir}-1)!}\ \frac{e_{ir}}{\mu_{ir}} \left[\frac{e_{ir}}{\mu_{ir}} \right]^{n_{ir}-1} \cdot$$

$$\prod_{\substack{j=1 \\ j\neq r}}^{R} \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}}$$

$$= \frac{n_i}{n_{ir}}\ e_{ir}\ f_i(n_{i1},n_{i2},\ldots,n_{ir}-1,\ldots,n_{iR})$$

If instead the load dependency function has the following composite form

$$F_{ir}(n_{i1},n_{i2},\ldots,n_{iR}) = g_{ir}(n_{ir})\ h_i\left[\sum_{j=1}^{R} n_{ij} \right]$$

then for service centers of type 2 and 4 (for type 1 we have shown that $g_{ir}(n_{ir})=1$ and therefore it corresponds to the case considered in the previous derivation) we get

$\mu_{ir}\ g_{ir}(n_{ir})\ h_i(n_i)\ f_{ir}(n_{i1},n_{i2},\ldots,n_{iR})$

$$= \mu_{ir}\ g_{ir}(n_{ir})\ h_i(n_i)\ \frac{n_i!}{\prod_{k=1}^{n_i} h_i(k)}\ \prod_{j=1}^{R} \frac{1}{n_{ij}!} \left[\frac{e_{ij}}{\mu_{ij}} \right]^{n_{ij}} \frac{1}{\prod_{l=1}^{n_{ij}} g_{ij}(l)}$$

$$= \mu_{ir}\ g_{ir}(n_{ir})\ h_i(n_i)\ \frac{n_i\ (n_i-1)!}{h_i(n_i)\prod_{k=1}^{n_i-1} h_i(k)}\ \frac{1}{n_{ir}\ (n_{ir}-1)!}\ \frac{e_{ir}}{\mu_{ir}}\left[\frac{e_{ir}}{\mu_{ir}}\right]^{n_{ir}-1}\ .$$

$$\frac{1}{g_{ir}(n_{ir})\prod_{l=1}^{n_{ir}-1} g_{ir}(l)}\ \prod_{\substack{j=1\\ j\neq r}}^{R} \frac{1}{n_{ij}!}\left[\frac{e_{ij}}{\mu_{ij}}\right]^{n_{ij}} \frac{1}{\prod_{l=1}^{n_{ij}} g_{ij}(l)}$$

$$= \frac{n_i}{n_{ir}}\ e_{ir}\ f_{ir}(n_{i1}, n_{i2}, \ldots, n_{ir}-1, \ldots, n_{iR})$$

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